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A generalization of the Lax equation defined by an arbitrary anti-automorphism

Maria Przybylska†

Institute of Physics, Nicholas Copernicus University, Grudziądzka 5, 87–100 Toruń, Poland

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Abstract. We consider a class of evolution equations on the Lie group $GL(n, \mathbb{R})$ or any of its closed subgroups, built by means of an arbitrary anti-automorphism of the associative algebra of all real n-dimensional matrices $\mathbb{M}^{n\times n}$. The set of first integrals and a method of construction for a Hamiltonian subclass is shown. This subclass has a connection with the factorization problem. A certain application of a matrix evolution equation built by means of transposition, related to the existence of (2, 0)- and (0, 2)-type tensor invariants in the theory of dynamical systems, is found.

1. Introduction

This year, three decades have passed since the celebrated paper by Lax [9] was published. Lax's idea was to replace evolution equations of a system in question by other ones defined in a different state space and possessing certain useful properties. The original evolution equations of the system are described on a certain manifold $\mathcal{P}$ (state space) with finite dimension m, by

$$\dot{\boldsymbol{x}} = F(\boldsymbol{x}) \tag{1.1}$$

where $\boldsymbol{x} = (x^1, \dots, x^m) \in \mathcal{P}$ and $F(\boldsymbol{x})$ denotes a vector field on $\mathcal{P}$. In Lax's approach information about the dynamics of this system are obtained from the analysis of the operator-type evolution equation on a set $\mathcal{M}$ of n-dimensional matrices:

$$\dot{L} = [L, M] \qquad L(t_0) = L_0 \tag{1.2}$$

where $[\cdot, \cdot]$ denotes the commutator. Matrices L, M depend, by assumption, on the dynamical variables $x^1, \dots, x^m$ in such a way that the matrix equation (1.2) is equivalent to the system (1.1). It means that differential equations for quantities $x^1, \dots, x^m$ obtained from (1.2) have the same form as in the system (1.1). If we succeed in finding a passage from (1.1) to (1.2) we call (1.2) the Lax representation of equation (1.1). So far, the Lax representation have been found for a large number of systems with finitely or infinitely many degrees of freedom.

The existence of such representations is important for several reasons:

(1) a set of first integrals for the representation (1.2) is known,

(2) the set $\mathcal{M}$ often has a nice structure of a Lie algebra or of a Lie group of matrices and equation (1.2) is of Hamiltonian type. In addition, in many cases first integrals are pairwise in involution. This fact greatly simplifies the proof of the integrability of (1.2)

† E-mail address: mprzyb@phys.uni.torun.pl

and in consequence the proof of the integrability of the system (1.1). The famous Adler–Kostant–Symes (AKS) theorem [1,8,15] allows us to systematically construct Hamiltonian systems and a hierarchy of commuting first integrals. The deeper meaning of the AKS theorem was explained in terms of classical R matrices by Semenov-Tian-Shansky and Reyman [11, 12].

A natural question arises: how does one find a Lax representation for a given evolution equation? We are only able to construct the Lax representation in a systematic manner for a system, for which we know a (1, 1)-type tensor field with the following property: the Lie derivative of this tensor field along integral curves of the system vanishes [5, 6]. This fact is useful because there exists a few ways of constructing such tensor fields. This is, for instance, the case in bi-Hamiltonian systems [2] with one nonsingular two-form.

We now address the question: are there any other (more general) operator equations with properties analogous to the Lax equation, from which we can obtain information about the dynamics of (1.1)? In particular: are there equations associated with the vanishing of the Lie derivative of (0, 2)- or (2, 0)-type tensor fields along integral curves of considered systems?

One such class of generalizations of the Lax equation on an arbitrary associative algebra is due to Bogoyavlensky [3]. He investigated the equation

$$\dot{L} = L\tau(M) - ML \qquad L(t_0) = L_0 \tag{1.3}$$

where $\tau(\cdot)$ denotes any automorphism of the associative algebra. The Bogoyavlensky equation has many interesting properties, in particular, the set of first integrals [3] and the construction of the Hamiltonian subclass of equations of the form (1.3) [13] are known. But obtaining a representation of this type is a pure art and consists of finding a suitable change of variables.

In this paper, we will analyse another type of generalized Lax equation on a set $\mathcal{G}$ with the structure of the Lie group $GL(n, \mathbb{R})$ or any of its closed subgroups, namely:

$$\dot{L} = -(L\kappa(M) + ML) \qquad L(t_0) = L_0 \in \mathcal{G} \tag{1.4}$$

where $M = M(t, L)$ is Lipschitz continuous and M belongs to the Lie algebra $\mathfrak{g}$ associated with the Lie group $\mathcal{G}$, and $\kappa(\cdot)$ is an arbitrary anti-automorphism of the associative algebra of all real n-dimensional matrices $\mathbb{M}^{n\times n}$. In addition, we assume that for every $M \in \mathfrak{g}$, $\kappa(M) \in \mathfrak{g}$.

The analysis of this type of matrix differential equation can be useful because there exist dynamical systems for which the representation (1.4) is easier to find than the representation with an automorphism (1.3). In addition, even if both representations exist for a certain dynamical system, it happens that the representation of the form (1.4) gives more information about the dynamics than (1.3). In the last section, we illustrate these statements with examples of (1.3) and (1.4) of the following form:

$$\dot{L} = LM - ML \qquad L(t_0) = L_0, \tag{1.5}$$

$$\dot{L} = -LM^T - ML \qquad L(t_0) = L_0 \tag{1.6}$$

where T denotes the transposition of matrices.

The organization of this paper is as follows. We begin the next section with introductory definitions and with an analysis of the general properties of equation (1.4). In section 3, the construction of a Hamiltonian class for equation (1.4) and a connection with the factorization problem is presented. In the final section, we analyse the transposition as a special case of an anti-automorphism transformation. Equation (1.4) with $\kappa(M) = M^T$, where T denotes the transposition of matrices, has a close connection with (0, 2)- and (2, 0)-type tensor invariants. In addition, in the last section we present dynamical systems representable in the form (1.6) and we comment on certain relations between representations of the type (1.5) and (1.6).

2. Basic notions and general properties of equations with an anti-automorphism

Let us recall that an anti-automorphism (of the second order) κ of the associative algebra of all n-dimensional real matrices $\mathbb{M}^{n\times n}$ is a map $\kappa : \mathbb{M}^{n\times n} \to \mathbb{M}^{n\times n}$ which fulfils the following conditions:

$$\begin{aligned} &\forall X_1, X_2 \in \mathbb{M}^{n\times n} \qquad \kappa(X_1 + X_2) = \kappa(X_1) + \kappa(X_2) \\ &\forall X_1, X_2 \in \mathbb{M}^{n\times n} \qquad \kappa(X_1 \cdot X_2) = \kappa(X_2) \cdot \kappa(X_1) \\ &\forall X \in \mathbb{M}^{n\times n} \qquad \kappa^2(X) = X. \end{aligned} \tag{2.1}$$

The second condition means that for an arbitrary nonsingular element $X \in \mathbb{M}^{n\times n}$

$$\kappa(X^{-1}) = \kappa^{-1}(X). \tag{2.2}$$

In addition, we assume

$$\forall M \in \mathfrak{g} \qquad \kappa(M) \in \mathfrak{g}. \tag{2.3}$$

We have assumed, that the evolution $t \to L(t)$ described by

$$\dot{L} = -(L\kappa(M) + ML) \qquad L(t_0) = L_0 \tag{2.4}$$

where $\kappa(\cdot)$ is an arbitrary anti-automorphism of $\mathbb{M}^{n\times n}$ fulfilling (2.3), is defined in the set $\mathcal{G}$ of nonsingular $n \times n$ real matrices. This limitation is important otherwise the existence of first integrals of (2.4) would not be guaranteed. We do not impose this assumption on the n-dimensional matrix M. In general, explicit time dependence and the dependence of M on L is allowed.

The important properties of equation (2.4) are formulated in the following theorems.

Theorem 1. *If the evolution is described by (2.4), then eigenvalues of the matrix $L\kappa^{-1}(L)$ are first integrals, or equivalently*

$$C_k = \mathrm{Tr}\{L\kappa^{-1}(L)\}^k \qquad k = 1, 2, \ldots \tag{2.5}$$

are first integrals of (2.4).

Proof. Using (2.4) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\{L\kappa^{-1}(L)\} = [L\kappa^{-1}(L), M]. \tag{2.6}$$

The above equation has the Lax form and this observation completes the proof. □

We see that first integrals given by (2.5) only exist if the evolution of the matrix $L(t)$ takes place in the set $\mathcal{G}$ of nonsingular matrices.

Theorem 2. *Assume that for the evolution equation (1.1) we have found two representations of the form (2.4):*

$$\dot{L}_1 = -(L_1\kappa(M_1) + M_1L_1) \qquad \dot{L}_2 = -(L_2\kappa(M_2) + M_2L_2) \tag{2.7}$$

where L_1, M_1 and L_2, M_2 are n_1- and n_2-dimensional matrices, respectively, and an anti-automorphism fulfils the condition

$$\forall X_1, X_2 \in \mathbb{M}^{n\times n} \qquad \kappa(X_1 \otimes X_2) = \kappa(X_1) \otimes \kappa(X_2) \tag{2.8}$$

where $\otimes$ denotes the Kronecker product of matrices [14]. We can then construct the new representation of the same form in the following manner:

$$L_3 = L_1 \otimes L_2 \qquad M_3 = M_1 \otimes \mathbb{1}_{n_2} + \mathbb{1}_{n_1} \otimes M_2 \tag{2.9}$$

where $\mathbb{1}_{n_1}$ and $\mathbb{1}_{n_2}$ denote n_1- and n_2-dimensional unity matrices, respectively.

Proof. We calculate $\dot{L}_3$ using (2.7) and $-L_3\kappa(M_3) - M_3L_3$ by means of (2.9) and the basic relation of the Kronecker product [14]:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD). \tag{2.10}$$

□

We can also generate integrals of motion I_{ij} from this new representation:

$$I_{ij} = \mathrm{Tr}\{[L_1\kappa^{-1}(L_1)]^i \otimes [L_2\kappa^{-1}(L_2)]^j\} = \mathrm{Tr}\{[L_1\kappa^{-1}(L_1)]^i\}\,\mathrm{Tr}\{[L_2\kappa^{-1}(L_2)]^j\} \tag{2.11}$$

where we used the relation [14]:

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B). \tag{2.12}$$

This method of generating a new representation is also valid in the case when $L_1 = L_2$. This means that one representation of the form (2.4) is sufficient to generate an infinite sequence of new representations. The described construction is analogous to the construction for the Lax representation [10]. In [10], matrices L_3, M_3 have the following forms:

$$L_3 = \alpha_1(L_1 \otimes L_2) + \alpha_2(\mathbb{1}_{n_1} \otimes L_2 + L_1 \otimes \mathbb{1}_{n_2}) \qquad M_3 = M_1 \otimes \mathbb{1}_{n_2} + \mathbb{1}_{n_1} \otimes M_2 \tag{2.13}$$

where α_1, α_2 are arbitrary real parameters. We see that the difference between the construction for the Lax representation and for the one described above consists of the absence of the second term in the expression for L_3. The existence of the second term is probably only characteristic for the Lax representation.

The subsequent properties are associated with the presence of a Lie group and Lie algebra structures. We assumed that the state space $\mathcal{G}$ is the Lie group $GL(n, \mathbb{R})$ or any of its closed subgroups. Matrices M are the elements of the Lie algebra $\mathfrak{g}$ associated with the Lie group $\mathcal{G}$.

Prior to formulating the next theorem we recall the definition of an anti-action of a Lie group on itself and the definition of an orbit. By an anti-action of a Lie group $\mathcal{G}$ on itself we mean the map $\rho : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $(Q, L_0) \mapsto \rho_Q(L_0)$ which fulfils the following properties:

$$\begin{aligned} &\forall Q_1, Q_2, L_0 \in \mathcal{G} \qquad \rho_{Q_1 \cdot Q_2}(L_0) = \rho_{Q_2} \circ \rho_{Q_1}(L_0) \\ &\forall Q, L_0 \in \mathcal{G} \qquad \rho_{Q^{-1}}(L_0) = \rho_Q^{-1}(L_0) \end{aligned} \tag{2.14}$$

where $\circ$ denotes the superposition of maps. Another term for an anti-action of $\mathcal{G}$ is the right action of $\mathcal{G}$ [4]. If we fix an element Q from $\mathcal{G}$, then we denote the anti-action of $\mathcal{G}$ on itself by means of a map $\rho_Q : \mathcal{G} \to \mathcal{G}$. An orbit $\mathcal{O}_{L_0}$ of an element L_0 is the set

$$\mathcal{O}_{L_0} = \{\rho_Q(L_0) | Q \in \mathcal{G}\}. \tag{2.15}$$

We can now formulate the following theorem.

Theorem 3. *Integral curves of (2.4) belong to orbits of the following anti-action of the Lie group $\mathcal{G}$ on itself:*

$$\forall Q, L_0 \in \mathcal{G} \qquad \rho_Q(L_0) := Q^{-1}L_0\kappa(Q^{-1}). \tag{2.16}$$

Proof. The above statement means, that for any $t \geqslant t_0$ there exists a suitable matrix $Q(t) \in \mathcal{G}$, such that

$$L(t) = Q(t)^{-1}L_0\kappa(Q(t)^{-1}). \tag{2.17}$$

Now, we need to associate any element M from $\mathfrak{g}$ with the element Q from the group $\mathcal{G}$. Any element $M \in \mathfrak{g}$ determines the corresponding element $Q \in \mathcal{G}$ by means of the standard relation (the well known Lie theorem):

$$\dot{Q} = QM \qquad Q(t_0) = \mathbb{1}. \tag{2.18}$$

Writing (1.4) we have indicated that an explicit dependence of the matrix M on t and the dependence on L are admissible. These dependences do not destroy the relation between the Lie group $\mathcal{G}$ and the Lie algebra $\mathfrak{g}$ if $M(t)$ is a continuous function of t and if $M(t)$ remains all the time in $\mathfrak{g}$. This important generalization of the Lie theorem is due to Watkins [16]. Now, if we calculate the time derivative of L (2.17) with respect to t using (2.18) we obtain (2.4). □

In the framework of Lie group structures, we can see the geometric meaning of first integrals described by theorem 1, if we introduce the notion of $\mathcal{G}$-invariant function. A function is $\mathcal{G}$-invariant with respect to the anti-action ρ of $\mathcal{G}$ on itself iff

$$\forall Q, L \in \mathcal{G} \qquad f(\rho_Q(L)) = f(L). \tag{2.19}$$

We denote by $I(\mathcal{G})$ a set of $\mathcal{G}$-invariant functions.

Theorem 4. *First integrals described by theorem 1 are $\mathcal{G}$-invariant functions with respect to the anti-action (2.16) of $\mathcal{G}$ on itself.*

Proof. Using (2.17) we obtain

$$L\kappa(L^{-1}) = Q^{-1}L_0\kappa(L_0^{-1})Q. \tag{2.20}$$

It is obvious from (2.20) that the expression $L\kappa^{-1}(L)$ changes in time by means of a similarity transformation and its eigenvalues do not change. □

3. The construction of Hamiltonian equations containing an arbitrary anti-automorphism and a connection with factorization problem

In this section, we describe a method for constructing Hamiltonian equations of the form (2.4) and a connection of this type of equations with the factorization problem of the Lie algebra $gl(n, \mathbb{R})$ and of the associated Lie group $gl(n, \mathbb{R})$. In the whole section we assume that $\mathcal{G} = GL(n, \mathbb{R})$. The presented method of constructing Hamiltonian-type equations may be used if in the Lie algebra $\mathfrak{g}$ there exists a classical R matrix [11,12] with special properties.

The construction of the Hamiltonian equations consists of giving the definition of a Poisson bracket and specifying a generating function, $\varphi \in C^\infty(\mathcal{G})$, which is interpreted as Hamiltonian.

First we define the Poisson bracket. The Lie algebra, $\mathfrak{g}$, of matrices carries the natural and in general degenerate indefinite scalar product $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$:

$$\forall M_1, M_2 \in \mathfrak{g} \qquad (M_1, M_2) := \mathrm{Tr}(M_1 M_2) \tag{3.1}$$

which is invariant with respect to the adjoint representation of the Lie algebra $\mathfrak{g}$, i.e.

$$\forall M_1, M_2 \in \mathfrak{g} \qquad (M_1, [M_2, M_3]) = -([M_2, M_1], M_3). \tag{3.2}$$

Using the product (3.1), we define for arbitrary $\psi \in C^\infty(\mathcal{G})$ the left- and the right-gradient $\mathrm{D}\psi(L), \mathrm{D}'\psi(L) \in \mathfrak{g}$ [12] by the relations

$$(\mathrm{D}\psi(L), M) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \psi(\mathrm{e}^{tM}L) \qquad (\mathrm{D}'\psi(L), M) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \psi(L\mathrm{e}^{tM}) \tag{3.3}$$

where $L \in \mathcal{G}$ and $M \in \mathfrak{g}$. The above definitions are connected with the notion of the gradient of an arbitrary matrix function $\psi \in C^\infty(\mathcal{G})$, $\mathrm{grad}\psi(L) \in \mathfrak{g}$ associated with the product (3.1), which was introduced as

$$\forall M \in \mathfrak{g} \qquad (\mathrm{grad}\psi(L), M) = \mathrm{d}\psi(L)M \tag{3.4}$$

where the right-hand side represents the value of the differential of the function on a tangent vector. The relations between left- $\mathrm{D}\psi(L)$ and right-$\mathrm{D}'\psi(L)$ and the gradient $\mathrm{grad}\psi(L)$ are the following:

$$\mathrm{D}\psi(L) = L\mathrm{grad}\psi(L) \qquad \mathrm{D}'\psi(L) = \mathrm{grad}\psi(L)L. \tag{3.5}$$

We assume that a classical R matrix fulfils the 'unitarity' condition (skew-symmetry condition):

$$\forall M_1, M_2 \in \mathfrak{g} \qquad (M_1, R(M_2)) = -(R(M_1), M_2). \tag{3.6}$$

We can now define a bracket $\{\cdot,\cdot\}_S$ [13] for $\psi_1, \psi_2 \in C^\infty(\mathcal{G})$:

$$\{\psi_1(L), \psi_2(L)\}_S := \tfrac{1}{2}[(R(\mathrm{D}\psi_1(L)), \mathrm{D}\psi_2(L)) - (R(\mathrm{D}'\psi_1(L)), \mathrm{D}'\psi_2(L))]. \tag{3.7}$$

This bracket is known as the Sklyanin bracket. Its properties are described by the following theorem.

Theorem 5. *(E.g. Semenov-Tian-Shansky [13].) If the classical R matrix fulfils the 'unitarity condition', then the Sklyanin bracket is a Poisson bracket.*

The proof of linearity and antisymmetry is trivial and the Jacobi identity may be checked analogously as in [13] or as in [7].

We can now write the Hamiltonian equation for any function $\psi \in C^\infty(\mathcal{G})$, generated by another arbitrary function $\varphi \in C^\infty(\mathcal{G})$:

$$\frac{\mathrm{d}\psi(L)}{\mathrm{d}t} = \{\varphi(L), \psi(L)\}_S. \tag{3.8}$$

The explicit form of this equation, obtained by means of the Sklyanin bracket and of the indefinite product (3.1), is the following:

$$(\mathrm{grad}\psi(L), \dot{L}) = \tfrac{1}{2}(R(\mathrm{D}\varphi(L)), \mathrm{D}\psi(L)) - \tfrac{1}{2}(R(\mathrm{D}'\varphi(L)), \mathrm{D}'\psi(L)). \tag{3.9}$$

Then, using the explicit form of $(\cdot,\cdot)$ and the properties of trace we can rewrite (3.9) as

$$\mathrm{Tr}(\mathrm{grad}\psi(L)\dot{L}) = \tfrac{1}{2}\,\mathrm{Tr}(\mathrm{grad}\psi(L)R(L\mathrm{grad}\varphi(L))L) - \tfrac{1}{2}\,\mathrm{Tr}(\mathrm{grad}\psi(L)LR(\mathrm{grad}\varphi(L)L)). \tag{3.10}$$

From this equation we obtain the evolution equation for L:

$$\dot{L} = \tfrac{1}{2}R(L\mathrm{grad}\varphi(L))L - \tfrac{1}{2}LR(\mathrm{grad}\varphi(L)L) \tag{3.11}$$

for an arbitrary function $\varphi \in C^\infty(\mathcal{G})$. There exists a set of functions and a subset of anti-automorphisms, for which equation (3.11) has the form

$$\dot{L} = -(L\kappa(M) + ML) \qquad L(t_0) = L_0. \tag{3.12}$$

Theorem 6. *Assume that the generating function φ belongs to the set $I(\mathcal{G})$ of $\mathcal{G}$-invariants of the anti-action (2.16) and an anti-automorphism κ is orthogonal with respect to the product (3.1)*

$$\forall M_1, M_2 \in \mathfrak{g} \qquad (\kappa(M_1), M_2) = (M_1, \kappa(M_2)) \tag{3.13}$$

and fulfils the commutation relation with the 'unitary' classical R matrix. Then the evolution equation (3.11) transforms to the form (3.12) with the matrix M of the form

$$M = -\tfrac{1}{2}R(L\mathrm{grad}\varphi(L)). \tag{3.14}$$

In addition, all functions from $I(\mathcal{G})$ are in involution with respect to the Sklyanin bracket.

As elements from $I(\mathcal{G})$ we can take, for example, the eigenvalues of the matrix $L\kappa(L^{-1})$.

Proof. In our case the condition of $\mathcal{G}$-invariance of a function $\varphi(L)$ with respect to the anti-action (2.16) reads

$$\forall Q, L \in \mathcal{G} \qquad \varphi(Q^{-1}L\kappa(Q^{-1})) = \varphi(L). \tag{3.15}$$

The infinitesimal version of the $\mathcal{G}$-invariance condition is the following

$$(\mathrm{grad}\varphi(L), ML + L\kappa(M)) = 0. \tag{3.16}$$

Using the explicit form of the product and properties of the trace we can rewrite this condition in the equivalent form

$$\forall L \in \mathcal{G} \qquad \forall M \in \mathfrak{g} \quad \mathrm{Tr}(\mathrm{grad}\varphi(L)ML) = -\,\mathrm{Tr}(\mathrm{grad}\varphi(L)L\kappa(M)). \tag{3.17}$$

Now, let us look at (3.10). The second term on the right-hand side can be transformed using the infinitesimal version of the $\mathcal{G}$-invariance condition for

$$M = \kappa(R(\mathrm{grad}\psi(L)L)) \tag{3.18}$$

as well as the assumed properties of the classical R matrix and of the anti-automorphism κ:

$$\begin{aligned} -\,\mathrm{Tr}(\mathrm{grad}\psi(L)LR(\mathrm{grad}\varphi(L)L)) &= \mathrm{Tr}(R(\mathrm{grad}\psi(L)L)\mathrm{grad}\varphi(L)L) \\ &= \mathrm{Tr}(LR(\mathrm{grad}\psi(L)L)\mathrm{grad}\varphi(L)) = -\,\mathrm{Tr}(\kappa(R(\mathrm{grad}\psi(L)L))L\mathrm{grad}\varphi(L)) \\ &= -\,\mathrm{Tr}(R(\mathrm{grad}\psi(L)L)\kappa(L\mathrm{grad}\varphi(L))) = \mathrm{Tr}(\mathrm{grad}\psi(L)L\kappa(R(L\mathrm{grad}\varphi(L)))). \end{aligned} \tag{3.19}$$

After these manipulations we obtain

$$\dot{L} = \tfrac{1}{2}R(L\mathrm{grad}\varphi(L))L + \tfrac{1}{2}L\kappa(R(L\mathrm{grad}\varphi(L))). \tag{3.20}$$

If we introduce the matrix M, defined by (3.14), then this equation transforms into

$$\dot{L} = -L\kappa(M) - ML \qquad M = -\tfrac{1}{2}R(L\mathrm{grad}\varphi(L)). \tag{3.21}$$

The second part of the theorem is immediately obtained for $\varphi_i, \varphi_j \in I(\mathcal{G})$ from the infinitesimal version of $\mathcal{G}$-invariance conditions for both of them. First, we use the $\mathcal{G}$-invariance condition for φ_i and put

$$M = \tfrac{1}{2}\kappa(R(\mathrm{grad}\varphi_j(L)L)). \tag{3.22}$$

Next, we take the $\mathcal{G}$-invariance condition for φ_j and put

$$M = \tfrac{1}{2}\kappa(R(L\mathrm{grad}\varphi_i(L))). \tag{3.23}$$

The sketch of explicit calculations is the following:

$$\begin{aligned} \{\varphi_i, \varphi_j\}_S &= \tfrac{1}{2}\,\mathrm{Tr}(\mathrm{grad}\varphi_j(L)R(L\mathrm{grad}\varphi_i(L))L) - \tfrac{1}{2}\,\mathrm{Tr}(\mathrm{grad}\varphi_j(L)LR(\mathrm{grad}\varphi_i(L)L)) \\ &= \tfrac{1}{2}\,\mathrm{Tr}(\mathrm{grad}\varphi_j(L)R(L\mathrm{grad}\varphi_i(L))L) + \tfrac{1}{2}\,\mathrm{Tr}(\mathrm{grad}\varphi_j(L)L\kappa(R(L\mathrm{grad}\varphi_i(L))))=0. \end{aligned} \tag{3.24}$$

□

Equation (3.21) has a connection with the factorization problem of the Lie algebra $\mathfrak{g}$ and of the Lie group $\mathcal{G}$ [11–13].

We denote by $\mathfrak{g}_R$ a Lie algebra with the underlying vector space the same as that of $\mathfrak{g}$ but with another Lie bracket $[\cdot,\cdot]_R : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$:

$$[M_1, M_2]_R = \tfrac{1}{2}([R(M_1), M_2] + [M_1, R(M_2)]. \tag{3.25}$$

We now define, using the classical R matrix, two Lie algebra homomorphisms $R\pm : \mathfrak{g}_R \to \mathfrak{g}$,

$$R_+ := \tfrac{1}{2}(R + \mathbb{1}) \qquad R_- := \tfrac{1}{2}(R - \mathbb{1}). \tag{3.26}$$

Note that $R_+ - R_- = \mathbb{1}$ and an arbitrary element $M \in \mathfrak{g}$ admits a unique decomposition:

$$M = M_+ - M_- \qquad \text{where} \quad M_+ := R_+(M), M_- := R_-(M). \tag{3.27}$$

In most cases classical R matrices are constructed as follows. Assume that there is a vector space decomposition of $\mathfrak{g}$ into a direct sum $\oplus$ of two subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$,

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-. \tag{3.28}$$

Let P_+ and P_- be projectors onto $\mathfrak{g}_+$ and $\mathfrak{g}_-$, respectively, parallel to the complementary subalgebra. Using the projectors we can construct a classical R matrix as

$$R := P_+ - P_-. \tag{3.29}$$

It is obvious that

$$R_+ = P_+ \qquad R_- = -P_- \tag{3.30}$$

and we can call (3.27) a factorization of the Lie algebra $\mathfrak{g}$.

Using this decomposition and the property of $\mathcal{G}$-invariance of the function $\varphi(L)$ it is convenient to rewrite (3.21) in two equivalent forms:

$$\dot{L} = -L\kappa(M_+) - M_+L \qquad \dot{L} = -L\kappa(M_-) - M_-L \tag{3.31}$$

where M_+, M_- are defined as

$$M_+ := -R_+(L\mathrm{grad}\varphi(L)) \qquad M_- := -R_-(L\mathrm{grad}\varphi(L)). \tag{3.32}$$

We now turn to the corresponding factorization of the Lie group $\mathcal{G}$. Let $\mathcal{G}$ and $\mathcal{G}_R$ be local Lie groups corresponding to $\mathfrak{g}$ and $\mathfrak{g}_R$, respectively. There are homomorphisms $R_\pm : \mathcal{G}_R \to \mathcal{G}$ which correspond to the Lie algebra homomorphisms $R_\pm$. We use the same letters denoting the Lie algebra and the Lie group homomorphisms. Each $Q \in \mathcal{G}$ admits a unique decomposition:

$$Q(t) = Q_+(t)Q_-^{-1}(t) \qquad \text{where} \quad Q_+(t) := R_+(Q(t)) \quad Q_-(t) := R_-(Q(t)). \tag{3.33}$$

We describe the relation of equations (3.31) with the factorization problem in the following theorem.

Theorem 7. *Let $Q_+(t)$ and $Q_-(t)$ be solutions of the following differental equations:*

$$\begin{aligned} \dot{Q}_+(t) &= Q_+(t)M_+(t) \qquad Q_+(t_0) = 1 \\ \dot{Q}_-(t) &= Q_-(t)M_-(t) \qquad Q_-(t_0) = 1 \end{aligned} \tag{3.34}$$

where M_+, M_- have forms (3.32) with the $\mathcal{G}$-invariant function $\varphi_k(L)$ of the form

$$\varphi_k(L) = \mathrm{Tr}(L\kappa(L^{-1}))^k \qquad k = 1, 2, \ldots, \tag{3.35}$$

where κ is any orthogonal anti-automorphism commuting with the 'unitary' classical R matrix. Then the integral curves of (3.21) are given by

$$L(t) = Q_+^{-1}L_0\kappa(Q_+^{-1}) = Q_-^{-1}L_0\kappa(Q_-^{-1}) \tag{3.36}$$

and $Q_+(t)$ and $Q_-(t)$ are solutions of the factorization problem (3.33) with the left-hand side given by

$$Q(t) = \exp\{-tL_0\mathrm{grad}\varphi_k(L_0)\} \qquad k = 1, 2, \ldots. \tag{3.37}$$

Proof. If we differentiate (3.36) with respect to t, then we obtain equations (3.31), which are equivalent to (3.21). In order to prove the second part of the theorem we only need evolution equations for $Q_+(t)$, $Q_-(t)$ (3.34), the equation of the decomposition of the Lie algebra (3.27), and the law of evolution for $\mathrm{grad}\varphi_k(L(t))$ for $\varphi_k(L)$ of the form (3.35). We now find this law of evolution. Using the explicit form of $\mathcal{G}$-invariant functions (3.35) we can calculate

$$\mathrm{grad}(\mathrm{Tr}(L\kappa(L^{-1}))^k) = k[(\kappa(L^{-1})L)^{k-1}\kappa(L^{-1}) - (L^{-1}\kappa(L))^k L^{-1}] \qquad k = 1, 2, \ldots. \tag{3.38}$$

Now, if we take (3.36) and the above relation we obtain the law of evolution for any function $\varphi_k(L)$ from the set (3.35) in the following form:

$$\mathrm{grad}\varphi_k(L(t)) = \kappa(Q_+(t))\mathrm{grad}\varphi_k(L_0)Q_+(t). \tag{3.39}$$

Finally, we present the second part of the proof:

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t} &= \dot{Q}_+Q_-^{-1} + Q_+(Q_-^{-1})^{\cdot} = Q_+(M_+ - M_-)Q_-^{-1} = Q_+(-L\mathrm{grad}\varphi_k(L))Q_+^{-1}Q_+Q_-^{-1}\\ &= -Q_+[Q_+^{-1}L_0\kappa(Q_+^{-1})][\kappa(Q_+)\mathrm{grad}\varphi_k(L_0)Q_+]Q_+^{-1}Q_+Q_-^{-1}\\ &= -L_0\mathrm{grad}\varphi_k(L_0)Q.\end{aligned} \tag{3.40}$$

Using the initial condition $Q(t_0) = 1$ we obtain (3.37). □

4. Evolution equations associated with transposition of matrices

In this section, we consider matrix equations associated with a special choice of an anti-automorphism—the transposition of matrices:

$$\dot{L} = -LM^T - ML \qquad L(t_0) = L_0. \tag{4.1}$$

In general, we can choose the state space as the set of all nonsingular matrices.

Eigenvalues of the matrix $L(L^{-1})^T$ form the set of first integrals. We observe that in cases when the matrix L is symmetric or antisymmetric for any $t \geqslant t_0$, all these first integrals equal 1 or -1, respectively. We deal with one of these two cases if we take the initial condition L_0 to be any symmetric or, respectively, antisymmetric matrix.

We note that if M is an antisymmetric matrix then (4.1) transforms into the Lax equation

$$\dot{\tilde{L}} = \tilde{L}\tilde{M} - \tilde{M}\tilde{L} \qquad \tilde{L}(t_0) = \tilde{L}_0. \tag{4.2}$$

However, in general the relation between the existence of the representation (4.1) and the Lax representation (4.2) is more complicated. A few simple examples, below, show that there are dynamical systems for which it is easy to construct the representation (4.1) but it is not clear how to construct the Lax representation with matrices $\tilde{L}$ and $\tilde{M}$ of the same dimension and structure as L and M in (4.1). We cannot exclude neither the existence nor the lack of the representation of the form (4.2) because, to our best knowledge, no theorem on global Lax representations for arbitrary dynamical systems exists. We only want to point out that certain dynamical systems are naturally representable in the form (4.1) with simple matrices L and M, whose elements are linear combinations of the variables of the dynamical system in question. For examples representable in the form (4.1) we were not able to find any nontrivial Lax representation with $\tilde{L}$ and $\tilde{M}$ with the same dimension and structure as L and M in (4.1). The MAPLE system proved useful in finding the representations.

We begin with the following dynamical system

$$\begin{aligned}\dot{a} &= 2a(a-b)\\ \dot{b} &= 2b(a-b).\end{aligned} \tag{4.3}$$

This system has a simple representation with the anti-automorphism of the form (4.1) if we take matrices L and M to be

$$L = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \qquad M = \begin{pmatrix} b-a & 0 \\ 0 & b-a \end{pmatrix}. \tag{4.4}$$

From this representation we can obtain the first integral I

$$I = \frac{a}{b}. \tag{4.5}$$

We note that I has the form of a quotient of polynomials of variables a and b. This quotient form of first integrals is characteristic for the representation (4.1) and is different from the polynomial form of first integrals obtained by means of the Lax representation. If we now look for the representation (4.2) of (4.3) in the set of two-dimensional real matrices with elements which are linear combinations of the variables a and b, we obtain many families of solutions. But most of these representations reconstruct only one equation of the system (4.3). For example, the representation with $\tilde{L}$ and $\tilde{M}$ of the form

$$\tilde{L} = \begin{pmatrix} 0 & 0 \\ a_{31}b & 0 \end{pmatrix} \qquad \tilde{M} = \begin{pmatrix} -(c_{41}+2)b + (2-c_{42})a & 0 \\ -c_{31}b - c_{32}a & -c_{41}b - c_{42}a \end{pmatrix} \tag{4.6}$$

where $a_{31}, c_{31}, c_{32}, c_{41}, c_{42}$ are real parameters, only reproduces the evolution equation for b. In all representations which generate both equations in (4.3), e.g.

$$\begin{aligned} \tilde{L} &= \begin{pmatrix} -2\frac{a_{22}a_{32}}{c_{32}a_{22}-\alpha}b + \frac{1}{2}(\alpha - c_{32}a_{22})a & -a_{22}b + a_{22}a \\ -a_{32}b + a_{32}a & 2\frac{a_{22}a_{32}}{c_{32}a_{22}-\alpha}b + \frac{1}{2}(c_{32}a_{22}-\alpha)a \end{pmatrix} \\ \tilde{M} &= \begin{pmatrix} \frac{a_{22}c_{32}(2-c_{41})+\alpha(c_{41}+2)}{c_{32}a_{22}-\alpha}b + \frac{c_{32}(c_{32}a_{22}-\alpha)+a_{32}(2-c_{42})}{a_{32}}a & \frac{a_{22}(c_{32}^2a_{22}-c_{32}\alpha+4a_{32})}{a_{32}(c_{32}a_{22}-\alpha)}b - \frac{\alpha}{a_{32}}a \\ c_{32}b - c_{32}a & -c_{41}b - c_{42}a \end{pmatrix} \end{aligned} \tag{4.7}$$

where $a_{22}, a_{32}, c_{32}, c_{41}, c_{42}$ are real parameters and α is a solution of the following quadratic equation

$$x^2 - 2a_{22}c_{32}x + c_{32}^2a_{22}^2 + 4a_{22}a_{32} = 0 \tag{4.8}$$

both eigenvalues of the matrix $\tilde{L}$ are equal to zero.

A similar situation occurs for the next dynamical system

$$\begin{aligned} \dot{a} &= a(a+b-2c) \\ \dot{b} &= b(a+b-2c) \\ \dot{c} &= 2c(b-c). \end{aligned} \tag{4.9}$$

This system has a natural representation with the transposition (4.1),

$$L = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} \qquad M = \begin{pmatrix} c-a & 0 \\ 0 & c-b \end{pmatrix}. \tag{4.10}$$

Using MAPLE for the Lax representation of (4.9) in the set of two-dimensional matrices with linear combinations of elements a, b and c, we do not get any pair of matrices $\tilde{L}$ and $\tilde{M}$ that reconstruct all of the equations (4.9). Now, if we pass to the set of three-dimensional matrices with elements of the same linear structure, then we obtain a few representations that generate the whole system. But in all these cases matrices $\tilde{L}$ have only one degenerate eigenvalue equal to zero.

Next, we take a dynamical system which has the well known Lax representation. It is a two-dimensional Toda lattice

$$\begin{aligned} \dot{b}_1 &= -2a_1^2 \\ \dot{b}_2 &= 2a_1^2 \\ \dot{a}_1 &= a_1(b_1 - b_2). \end{aligned} \tag{4.11}$$

Its Lax representation is

$$\tilde{L} = \begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 \end{pmatrix} \qquad \tilde{M} = \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}. \tag{4.12}$$

This system is representable in the form of the matrix evolution equation (4.1) with the transposition as well, e.g. if we choose

$$L = \begin{pmatrix} b_1 - \frac{3}{2}a_1 & 2b_1 - b_2 + 2a_1 \\ b_1 - 2b_2 + 2a_1 & 4b_2 + 6a_1 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & \frac{a_1}{2} \\ -2a_1 & 0 \end{pmatrix}. \tag{4.13}$$

For this choice $L(L^T)^{-1}$ has two different eigenvalues.

The last example concerns the following dynamical system

$$\begin{aligned} \dot{a}_1 &= 2a_1^2 + b_1(2b_2 + b_1) + b_2^2 \\ \dot{a}_2 &= 2a_2^2 + b_2(2b_1 + b_2) + b_1^2 \\ \dot{b}_1 &= (a_1 + a_2)(2b_1 + b_2) \\ \dot{b}_2 &= (a_1 + a_2)(2b_2 + b_1) \end{aligned} \tag{4.14}$$

which is representable in the form (4.1), e.g. if we take L and M to be

$$L = \begin{pmatrix} a_1 & b_1 \\ b_2 & a_2 \end{pmatrix} \qquad M = \begin{pmatrix} -a_1 & -b_1 - b_2 \\ -b_1 - b_2 & -a_2 \end{pmatrix}. \tag{4.15}$$

Using MAPLE we are not able to find the Lax representation in the set of two-dimensional matrices with linear combinations of elements a_1, a_2, b_1, b_2 as, generating the whole system (4.14).

These examples show that there are dynamical systems for which representations with the transposition (4.1) are more natural and easier to find.

The equation

$$\dot{L} = -LM^T - ML \qquad L(t_0) = L_0 \tag{4.16}$$

has one more interesting property: it is related to special classes of tensor invariants of any autonomous dynamical system

$$\dot{x}^i = F^i(x^1, \ldots, x^m) \qquad i = 1, \ldots, m \tag{4.17}$$

on an m-dimensional manifold with coordinates $(x^1, \ldots, x^m)$. A tensor field T is a time-independent tensor invariant [2] of the dynamical system (4.17) iff

$$\mathcal{L}_F T = 0 \tag{4.18}$$

where $\mathcal{L}_F$ denotes the Lie derivative with respect to the vector field F. For our needs we only consider in detail the (0, 2)- and (2, 0)-type time-independent tensor invariants. Now, we write two formulae of the Lie derivative $\mathcal{L}_F$ of (2, 0)- and (0, 2)-type tensor fields Θ and Ψ, respectively, along the vector field $F(\boldsymbol{x})$:

$$\begin{aligned} (\mathcal{L}_F\Theta)^{ij} &= \sum_{k=1}^{m} \left[\frac{\partial \Theta^{ij}}{\partial x^k} F^k - \Theta^{ik} \frac{\partial F^j}{\partial x^k} - \Theta^{kj} \frac{\partial F^i}{\partial x^k} \right] \\ (\mathcal{L}_F\Phi)_{ij} &= \sum_{k=1}^{m} \left[\frac{\partial \Phi_{ij}}{\partial x^k} F^k + \Phi_{ik} \frac{\partial F^k}{\partial x^j} + \Phi_{kj} \frac{\partial F^k}{\partial x^i} \right]. \end{aligned} \tag{4.19}$$

It is obvious that if Θ and Φ are time-independent tensor invariants of the autonomous dynamical system generated by the field F, then the above equations transform into

$$\begin{aligned} \dot{\Theta}^{ij} &= \sum_{k=1}^{m} \left[\Theta^{ik} \frac{\partial F^j}{\partial x^k} + \frac{\partial F^i}{\partial x^k} \Theta^{kj} \right] \\ \dot{\Phi}_{ij} &= \sum_{k=1}^{m} \left[-\Phi_{ik} \frac{\partial F^k}{\partial x^j} - \frac{\partial F^k}{\partial x^i} \Phi_{kj} \right]. \end{aligned} \tag{4.20}$$

We introduce two matrices:

$$P^i_k := -\frac{\partial F^i}{\partial x^k} \qquad R^i_k := \frac{\partial F^k}{\partial x^i}. \tag{4.21}$$

The symbols P^i_k, R^i_k denote the elements in the ith rows and the kth columns of the matrices P and R, respectively. Now we can rewrite equations (4.20) as

$$\dot\Theta = -\Theta P^T - P\Theta \qquad \dot\Phi = -\Phi R^T - R\Phi. \tag{4.22}$$

We can then formulate the following theorem.

Theorem 8. *Evolution equations for time-independent* $(0, 2)$*- and* $(2, 0)$*-type tensor invariants of the autonomous dynamical system (4.17) have a matrix representation with the transposition (4.16).*

We can see a similarity to the Lax representation case, which is associated with the time-independent $(1, 1)$-type tensor invariant [5, 6].

This statement can be regarded as a proposal of the method for a systematic search for a matrix representation of type (4.16) for the autonomous system (4.17). Namely, if we find $(2, 0)$- or $(0, 2)$- type tensor field Θ or Φ, respectively, with vanishing Lie derivative along integral curves of (4.17), then we can construct a matrix representation of the type (4.16). We need only make the following substitutions:

(1) in the case of the existence of the $(2, 0)$-type time-independent tensor invariant Θ:

$$L^{ij} := \Theta^{ij} \qquad M^i_j := -\frac{\partial F^i}{\partial x^j} \tag{4.23}$$

(2) and in the case of the existence of the (0,2)-type time-independent tensor invariant Φ:

$$L_{ij} := \Phi_{ij} \qquad M^i_j := \frac{\partial F^j}{\partial x^i}. \tag{4.24}$$

In this manner, we transform the problem of looking for a matrix representation with the transposition (4.16) of the dynamical system (4.17) into the problem of looking for certain tensor invariants.

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